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Evenness conditions for four-factor cross-nested models

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SUMMARY

Fonseca et al. (2003) obtained UMVUE for the variance components of balanced cross nested models. The estimators were the difference of a positive and a negative part. Unbiased estimators are obtained for the variance components of such models with cross-nesting. Following Michalski & Zmyślony (1996) we may use the quotient of the positive by the negative part of the estimators to test the nullity of the variance components. If either the degrees of freedom in the numerator or in the denominator are even we have, (Fonseca et al., 2002) an exact expression for the distribution of the test statistic. It is thus interesting to see if this evenness conditions are a rarity or if they are satisfied in many circumstances. If we name as first evenness condition (1^{st}) that all components of the vector $\underline{g}_1$ are even and as second evenness condition (2^{nd}) that all components of the vector $\underline{g}_2$ are even, when at least one of these evenness conditions holds we have an exact expression for the distribution of the test statistic. We will answer this question for four factors models, showing that in more then half of the possible degrees of freedom combinations, at least one of the evenness conditions holds.

Key words: Cross nested balanced models, variance components, usual and generalized F tests, evenness conditions.

1. Introduction

For four-factors models we have the following possible cases:

(a) Fully nested models $[1 \supset 2 \supset 3 \supset 4]$.

(b) Two groups of nested factors that cross: there are two possibilities – in one of them, one group has three factors and the other one factor $[(1 \supset 2 \supset 3) \times 4]$, while in the other case both groups have two factors $[(1 \supset 2) \times (3 \supset 4)]$.

(c) Three groups of nested factors that cross: one group has two factors and the other two groups only one factor $\left[(1 \supset 2)\times 3\times 4\right]$.
(d) Four factors that cross $\left[1\times 2\times 3\times 4\right]$.

In what follows I_m will be the identity matrix of order m, J_m will be the matrix of order m with all components equal to 1 and $\underline{1}_m$ the vector with m components equal to 1. We represent by $E(\underline{U})$ and $\sum(\underline{U})$ the mean vector and covariance matrix of $\underline{U}$. We write $\underline{y} \sim N(\underline{\mu};\sigma^2 M)$, when $\underline{y}$ is normal with $E(\underline{y})=\underline{\mu}$ and $\sum(\underline{U})=\sigma^2 M$, and $S \sim \gamma\chi^2_m$ if S is the product by γ of a central chi-square with m degrees of freedom.

2. Estimators

Let L be the number of groups of factors. This will increase from $L=1$ when the model is fully nested to $L=4$ when the four factors cross.
The number of factors in the L groups will be $u_1,\dots,u_L$. In the l^{th} group the factors will have indexes $h_l=1,\dots,u_l$; we will write $h_l=0$ when no factor in the l^{th} group is considered, $l=1,\dots,L$.
The sets of factors belonging to distinct groups correspond to vectors $\underline{h}$. The set of these vectors will be

$$\Gamma=\{\underline{h}:h_l=0,\dots,u_l;l=1,\dots,L\}.$$

For the balanced cross-nested design we have the model

$$\underline{y}=\sum_{\underline{h}\in\Gamma} X(\underline{h})\underline{\beta}^{c(\underline{h})}(\underline{h})+\underline{e},$$

where $\beta(\underline{0})=\mu$, $\underline{\beta}^{c(\underline{h})}(\underline{h})\sim N\left(\underline{0}^{c(\underline{h})};\sigma^2(\underline{h})I_{c(\underline{h})}\right)$, $\underline{h}\in\Gamma$ are mutually independent and $\underline{e}\sim N\left(\underline{0};\sigma^2 I_n\right)$.

Moreover $X(\underline{0})=\underline{1}_n$, with $n=\left[\prod_{l=1}^{L}c_l(u_l)\right]\times r$, $X(\underline{h})=\left[\bigotimes_{l=1}^{L}X_l(h_l)\right]\otimes\underline{1}_r$, with $X_l(h_l)=I_{c_l(h_l)}\otimes\underline{1}_{b_l(h_l)}$, and $b_l(h_l)=\frac{c_l(u_l)}{c_l(h_l)}$, $h_l=0,\dots,u_l$, $l=1,\dots,L$, where $c_l(h_l)$ is the total number of levels of factor h_l in group l. With $a_l(0)=1$, if the first factor has $a_l(1)$ levels and the following have $a_l(h_l)$ levels nested inside each level of the preceding factor we will have $c_l(h_l)=\prod_{j=1}^{h_l}a_l(j)$, $l=1,\dots,L$.

It is straightforward to see that $E\left(\underline{y}^n\right)=\underline{1}^n\mu$ and
$\Sigma\left(\underline{y}^n\right)=\sum_{\underline{h}\in\Gamma}\sigma^2\left(\underline{h}\right)M\left(\underline{h}\right)+\sigma^2 I_n$, where $M\left(\underline{h}\right)=X\left(\underline{h}\right)X\left(\underline{h}\right)^T=b\left(\underline{h}\right)Q\left(\underline{h}\right)$, with
$Q\left(\underline{h}\right)=\left[\otimes_{l=1}^{L}Q_l\left(h_l\right)\right]\otimes\frac{1}{r}J_r$ the orthogonal projection matrices and
$b\left(\underline{h}\right)=\prod_{l=1}^{L}b_l\left(h_l\right)\times r$.

Then with

$$\begin{cases}Q_l^*\left(0\right)=Q_l\left(0\right)\\ Q_l^*\left(h_l\right)=Q_l\left(h_l\right)-Q_l\left(h_l-1\right)\end{cases},\quad l=1,\dots,L,$$

we have $M\left(\underline{h}\right)=b\left(\underline{h}\right)\sum_{\underline{k}\le\underline{h}}Q^*\left(\underline{k}\right)$, where the $Q^*\left(\underline{k}\right)=\left[\otimes_{l=1}^{L}Q_l^*\left(k_l\right)\right]\otimes\frac{1}{r}J_r$, are mutually orthogonal orthogonal projection matrices.

Now with $\bar{Q}=I_n-\sum_{\underline{k}\in\Gamma}Q^*\left(\underline{k}\right)$, according to the last expression for $M\left(\underline{h}\right)$, we can rewrite $\Sigma\left(\underline{y}^n\right)$ as

$$\Sigma\left(\underline{y}^n\right)=\sum_{\underline{k}\in\Gamma}\gamma(\underline{k})Q^*\left(\underline{k}\right)+\sigma^2\bar{Q},$$

with $\gamma(\underline{k})=\sigma^2+\sum_{\underline{h}:\underline{k}\le\underline{h}}b\left(\underline{h}\right)\sigma^2\left(\underline{h}\right)$.

We now establish

Proposition 1: $\sigma^2\left(\underline{h}\right)=\frac{1}{b\left(\underline{h}\right)}\sum_{\underline{h}'\in U\left(\underline{h}\right)}\left(-1\right)^{m\left(\underline{h},\underline{h}'\right)}\gamma\left(\underline{h}'\right)$, where $m\left(\underline{h},\underline{h}'\right)$ is the number of components of $\underline{h}$ that are less than the corresponding components of $\underline{h}'$, and $U\left(\underline{h}\right)=\left\{\underline{h}':h_i\le h_i'\le\min\left[\left(h_i+1\right);u_i\right],i=1,\dots,L\right\}$ means that the components of $\underline{h}'$ are not less then the components of $\underline{h}$ and they do not exceed these by more than one unit.

Proof: Starting from the second member of the equality we have .

$$\frac{1}{b\left(\underline{h}\right)}\sum_{\underline{h}'\in U\left(\underline{h}\right)}\left(-1\right)^{m\left(\underline{h},\underline{h}'\right)}\gamma\left(\underline{h}'\right)=\frac{1}{b\left(\underline{h}\right)}\sum_{\underline{h}'\in U\left(\underline{h}\right)}\left(-1\right)^{m\left(\underline{h},\underline{h}'\right)}\left[\sigma^2+\sum_{\left\{\underline{h}'':\underline{h}'\le\underline{h}''\right\}}b\left(\underline{h}''\right)\sigma^2\left(\underline{h}''\right)\right]$$

Therefore, $\sigma^2\left(\underline{h}''\right)$ has coefficients $(-1)^{m\left(\underline{h},\underline{h}'\right)}$ in the corresponding terms $\underline{h}'$ such as $\underline{h}' \in U\left(\underline{h}\right)$ and $\underline{h}' \leq \underline{h}''$. If $\underline{h}'' = \underline{h}$ we must have $\underline{h}' \leq \underline{h}$ and having only one term in $\sigma^2\left(\underline{h}''\right)$, it will have coefficient $(-1)^{m(\underline{h},\underline{h})} = 1$. If $\underline{h}'' \neq \underline{h}$ we will have $\underline{h}'' > \underline{h}$. Let $C\left(\underline{h}''\right) = \left\{ j : h_j < h_j'' \right\}$.
Since $C' \subseteq C\left(\underline{h}''\right)$ there exists $\underline{h}' \in U\left(\underline{h}\right)$ such as

$$\begin{cases} j \in C' & h_j' = h_j + 1 \\ j \notin C' & h_j' = h_j \end{cases}$$

and $\sigma^2\left(\underline{h}''\right)$ enter in the corresponding term of $\underline{h}'$ with coefficient $(-1)^{m\left(\underline{h},\underline{h}'\right)}$.

We will have $\begin{pmatrix} m\left(\underline{h},\underline{h}''\right) \\ l \end{pmatrix}$ sets $C' \subseteq C\left(\underline{h}''\right)$ with $\#\left(C'\right) = l$, so the coefficient of $\sigma^2\left(\underline{h}''\right)$ will be

$$\sum_{l=0}^{m\left(\underline{h},\underline{h}''\right)} \begin{pmatrix} m\left(\underline{h},\underline{h}''\right) \\ l \end{pmatrix} (-1)^l = 0. \qquad \blacksquare$$

Since $U\left(\underline{h}\right) = \left\{ \underline{h}' : h_i \leq h_i' \leq \min\left[\left(h_i + 1\right); u_i\right], i = 1, \ldots, L \right\}$ we can consider two subsets $U\left(\underline{h}\right)^+$ and $U\left(\underline{h}\right)^-$ constituted by $\underline{h}' \in U\left(\underline{h}\right)$ for which $m\left(\underline{h},\underline{h}'\right)$ is even or odd, respectively.
As

$$\gamma(\underline{h}) = \sigma^2 + \sum_{\underline{h}' : \underline{h} \leq \underline{h}' \leq \underline{u}} b\left(\underline{h}'\right)\sigma^2\left(\underline{h}'\right) = \sigma^2 + \sum_{\underline{h}' \in U\left(\underline{h}\right)^+} b\left(\underline{h}'\right)\sigma^2\left(\underline{h}'\right) + \sum_{\underline{h}' \in U\left(\underline{h}\right)^-} b\left(\underline{h}'\right)\sigma^2\left(\underline{h}'\right)$$

the variance components $\sigma^2\left(\underline{h}\right)$ are associated to sets of vectors belonging to different groups, and (Fonseca et al., 2003) the positive and negative parts of the UMVUE for $\sigma^2\left(\underline{h}\right)$ are

$$\begin{cases} \tilde{\sigma}^2\left(\underline{h}\right)^+ = \sum_{\underline{k} \in U\left(\underline{h}\right)^+} d\left(\underline{k}\right)\chi^2_{g\left(\underline{k}\right)} & ; \underline{h} < \underline{u} \\ \tilde{\sigma}^2\left(\underline{h}\right)^- = \sum_{\underline{k} \in U\left(\underline{h}\right)^-} d\left(\underline{k}\right)\chi^2_{g\left(\underline{k}\right)} & ; \underline{h} < \underline{u} \end{cases}.$$

The coefficients $d(\underline{k})$ may be found in Fonseca et al (Fonseca et al., 2003) but we are interested in the degrees of freedom $g(\underline{k}) = \text{rank}\left[Q^*(\underline{k})\right] = \prod_{l=1}^{L} g_l(k_l)$, $\underline{k} \in \Gamma$, with

$$\begin{cases} g_l(0) = 1 & ; l = 1,\ldots,L \\ g_l(1) = c_l(1) & ; l = 1,\ldots,L \\ g_l(k_l) = \text{rank}\left[Q^*(k_l)\right] = c_l(k_l) - c_l(k_l - 1) & ; k_l = 2,\ldots,u_l;\; l = 1,\ldots,L \end{cases}$$

We now point out that the number of terms in each part of the estimator is $2^{m(\underline{h},\underline{k})-1}$.

Moreover $S(\underline{h}) = \left\| Q^*(\underline{h})\underline{y} \right\|^2 \sim \gamma(\underline{h})\chi^2_{g(\underline{h})}$, $\underline{h} \neq \underline{0}$ and $S = \left\| \bar{Q}\underline{y} \right\|^2 \sim \sigma^2 \chi^2_g$, with $g = n - c(\underline{u})$, where these chi-squares are independents.

In the cases in which $m(\underline{h},\underline{u}) = 1$, there exist unbiased estimators for the variance components given by difference of two mean squares, so we can test the nullity hypothesis of these components using the usual F tests. Likewise $\tilde{\sigma}^2(\underline{u})$ will also be given by the difference of two mean squares.

Then we have the hypothesis $H_0 : \sigma^2(\underline{h}) = 0$ against $H_1 : \sigma^2(\underline{h}) > 0$. Now with

$$\theta(\underline{h}) = \frac{\gamma(\underline{h})}{\gamma(\underline{h}')},$$

where $\underline{h}'$ is $h_i' = h_i + 1$, $i = 1,\ldots,L$, these hypothesis can be rewritten as $H_0 : \theta(\underline{h}) = 1$ against $H_1 : \theta(\underline{h}) > 1$.

So, we can use the usual F tests with statistics

$$\Im(\underline{h}) = \frac{g(\underline{h}')}{g(\underline{h})}\frac{S(\underline{h})}{S(\underline{h}')} \sim \frac{g(\underline{h}')}{g(\underline{h})}\frac{\gamma(\underline{h})}{\gamma(\underline{h}')}\frac{\chi^2_{g(\underline{h})}}{\chi^2_{g(\underline{h}')}} = \theta(\underline{h})\frac{g(\underline{h}')}{g(\underline{h})}\frac{\chi^2_{g(\underline{h})}}{\chi^2_{g(\underline{h}')}}.$$

In the cases in which $m(\underline{h},\underline{u}) > 1$, there do not exist unbiased estimators for the variance components given by the difference of two mean squares. Thus it is impossible to obtain usual F tests to test the hypothesis $H_0 : \sigma^2(\underline{h}) = 0$ against $H_1 : \sigma^2(\underline{h}) > 0$. Now with

$$\theta(\underline{h}) = \frac{\sum_{\underline{h}' \in U(\underline{h})^+} \gamma(\underline{h}')}{\sum_{\underline{h}' \in U(\underline{h})^-} \gamma(\underline{h}')},$$

these hypotheses can be rewritten as $H_0 : \theta(\underline{h}) = 1$ against $H_1 : \theta(\underline{h}) > 1$.

Hence we can use generalized F tests with statistics

$$\Im(\underline{h})=\theta(\underline{h})=\frac{\sum_{\underline{h}'\in U(\underline{h})^+}\frac{S(\underline{h}')}{g(\underline{h}')}}{\sum_{\underline{h}'\in U(\underline{h})^-}\frac{S(\underline{h}')}{g(\underline{h}')}}\sim\frac{\sum_{\underline{h}'\in U(\underline{h})^+}\frac{\gamma(\underline{h}')\chi^2_{g(\underline{h}')}}{g(\underline{h}')}}{\sum_{\underline{h}'\in U(\underline{h})^-}\frac{\gamma(\underline{h}')\chi^2_{g(\underline{h}')}}{g(\underline{h}')}}=\theta(\underline{h})\frac{\sum_{\underline{h}'\in U(\underline{h})^+}\frac{\tau_1(\underline{h}')\chi^2_{g(\underline{h}')}}{g(\underline{h}')}}{\sum_{\underline{h}'\in U(\underline{h})^-}\frac{\tau_2(\underline{h}')\chi^2_{g(\underline{h}')}}{g(\underline{h}')}},$$

where $\tau_1(\underline{h}')$ and $\tau_2(\underline{h}')$ are the nuisance parameters for the numerator and the denominator, given by

$$\tau_1(\underline{h}')=\frac{\gamma(\underline{h}')}{\sum_{\underline{h}''\in U(\underline{h})^+}\gamma(\underline{h}'')},\text{ for } \underline{h}'\in U(\underline{h})^+$$

and

$$\tau_2(\underline{h}')=\frac{\gamma(\underline{h}')}{\sum_{\underline{h}''\in U(\underline{h})^-}\gamma(\underline{h}'')},\text{ for } \underline{h}'\in U(\underline{h})^-.$$

In the next section we will discuss in detail whether either evenness condition is met for these statistics when there are four factors.

3. Four-Factor Models

Now $\underline{g}_1$ $\left[\underline{g}_2\right]$ will be the vector whose components are the degrees of freedom of the $g(\underline{h})$, $\underline{h}\in U(\underline{h})$. The degrees of freedom $g(\underline{h})$ being even or odd will depend on the $a_l(h_l)$, $h_l=0,\dots,u_l$, $l=1,\dots,L$ being even or odd.

3.1. Fully nested model

Since $L=1$, the UMVUE will be the difference of two mean squares and we may use the usual F tests.

3.2. Two groups of nested factors that cross

3.2.1. The first factor nests the second that nests the third and crosses with the fourth factor.

Since $L=2$ we write $\sigma^2(h_1,h_2)$, $m(h_1,h_2)$ and $g(k_1,k_2)$. The cases in which $m(h_1,h_2)$ is larger than 1 are

$$m(1,0)=m(2,0)=2,$$

so these are the cases for which we study the evenness conditions. The last column of the following tables indicates which evenness conditions hold. These conditions' holding or not depends only on the number of levels:

(a) For $\sigma^2(1,0)$ we have eight possibilities according as $a_1(1)$, $a_1(2)$ and $a_2(1)$ are even or odd. In Table 1 we present these possibilities.

Table 1 – Evenness Conditions

$a_1(1)$	$a_1(2)$	$a_2(1)$	$g(1,0)$	$g(2,1)$	$g(2,0)$	$g(1,1)$	*Conditions*
Even	Even	Even	Odd	Even	Even	Odd	-----
Even	Even	Odd	Odd	Even	Even	Even	*2nd*
Even	Odd	Even	Odd	Even	Even	Odd	-----
Even	Odd	Odd	Odd	Even	Even	Even	*2nd*
Odd	Even	Even	Even	Odd	Odd	Even	-----
Odd	Even	Odd	Even	Even	Odd	Even	*1st*
Odd	Odd	Even	Even	Even	Even	Even	*1st, 2nd*
Odd	*Odd*	*Odd*	*Even*	*Even*	*Even*	*Even*	*1st, 2nd*

(b) For $\sigma^2(2,0)$ we have sixteen possibilities according as $a_1(1)$, $a_1(2)$, $a_1(3)$ and $a_2(1)$ are even or odd. In Table 2 we present these possibilities.

Table 2 – Evenness Conditions

$a_1(1)$	$a_1(2)$	$a_1(3)$	$a_2(1)$	$g(2,0)$	$g(3,1)$	$g(3,0)$	$g(2,1)$	Conditions
Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Even	Even	Even	Odd	Even	Even	Even	Even	1st, 2nd
Even	Even	Odd	Even	Even	Even	Even	Even	1st, 2nd
Even	Even	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Even	Odd	Even	Even	Even	Even	Even	Even	1st, 2nd
Even	Odd	Even	Odd	Even	Even	Even	Even	1st, 2nd
Even	Odd	Odd	Even	Even	Even	Even	Even	1st, 2nd
Even	Odd	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Even	Even	Even	Odd	Even	Even	Odd	-----
Odd	Even	Even	Odd	Odd	Even	Even	Even	2nd
Odd	Even	Odd	Even	Odd	Even	Even	Odd	-----
Odd	Even	Odd	Odd	Odd	Even	Even	Even	2nd
Odd	Odd	Even	Even	Even	Odd	Odd	Even	-----
Odd	Odd	Even	Odd	Even	Even	Odd	Even	1st
Odd	Odd	Odd	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Odd	Even	Even	Even	Even	1st, 2nd

3.2.2. The first factor nests the second and crosses with the third that nests the fourth factor.

Since $L=2$ we write $\sigma^2(h_1,h_2)$, $m(h_1,h_2)$ and $g(k_1,k_2)$. The cases in which $m(h_1,h_2)$ is larger than 1 are

$$m(1,0)=m(0,1)=m(1,1)=2,$$

so these are the cases for which we study the evenness conditions. The last column of the following tables indicates which evenness conditions hold. These conditions' holding or not depends only on the number of levels:

(a) For $\sigma^2(1,0)$ we have eight possibilities according as $a_1(1)$, $a_1(2)$ and $a_2(1)$ are even or odd. In Table 3 we present these possibilities.

Table 3 – Evenness Conditions

$a_1(1)$	$a_1(2)$	$a_2(1)$	$g(1,0)$	$g(2,1)$	$g(1,1)$	$g(2,0)$	Conditions
Even	Even	Even	Odd	Even	Odd	Even	-----
Even	Even	Odd	Odd	Even	Even	Even	2nd
Even	Odd	Even	Odd	Even	Odd	Even	-----
Even	Odd	Odd	Odd	Even	Even	Even	2nd
Odd	Even	Even	Even	Odd	Even	Odd	-----
Odd	Even	Odd	Even	Even	Even	Odd	1st
Odd	Odd	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Even	Even	Even	Even	1st, 2nd

(b) For $\sigma^2(0,1)$ we have eight possibilities according as $a_1(1)$, $a_2(1)$ and $a_2(2)$ are even or odd. In Table 4 we present these possibilities.

Table 4 – Evenness Conditions

$a_1(1)$	$a_2(1)$	$a_2(2)$	$g(0,1)$	$g(1,2)$	$g(1,1)$	$g(0,2)$	Conditions
Even	Even	Even	Odd	Even	Odd	Even	-----
Even	Even	Odd	Odd	Even	Odd	Even	-----
Even	Odd	Even	Even	Odd	Even	Odd	-----
Even	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Even	Even	Odd	Even	Even	Even	2nd
Odd	Even	Odd	Odd	Even	Even	Even	2nd
Odd	Odd	Even	Even	Even	Even	Odd	1st
Odd	Odd	Odd	Even	Even	Even	Even	1st, 2nd

(c) For $\sigma^2(1,1)$ we have sixteen possibilities according as $a_1(1)$, $a_1(2)$, $a_2(1)$ and $a_2(2)$ are even or odd. In Table 5 we present these possibilities.

Table 5 – Evenness Conditions

$a_1(1)$	$a_1(2)$	$a_2(1)$	$a_2(2)$	$g(1,0)$	$g(2,2)$	$g(1,2)$	$g(2,1)$	Conditions
Even	Even	Even	Even	Odd	Even	Even	Even	2nd
Even	Even	Even	Odd	Odd	Even	Even	Even	2nd
Even	Even	Odd	Even	Even	Even	Odd	Even	1st
Even	Even	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Even	Odd	Even	Even	Odd	Even	Even	Even	2nd
Even	Odd	Even	Odd	Odd	Even	Even	Even	2nd
Even	Odd	Odd	Even	Even	Even	Odd	Even	1st
Even	Odd	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Even	Even	Even	Even	Even	Even	Odd	1st
Odd	Even	Even	Odd	Even	Even	Even	Odd	1st
Odd	Even	Odd	Even	Even	Odd	Even	Even	2nd
Odd	Even	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Even	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Odd	Even	Even	Even	Even	1st, 2nd

3.3. Three groups of nested factors that cross

Since $L=3$ we write $\sigma^2(h_1,h_2,h_3)$, $m(h_1,h_2,h_3)$ and $g(k_1,k_2,k_3)$. The cases in which $m(h_1,h_2,h_3)$ is larger than 1 are

$$m(1,0,0)=3$$

and

$$m(0,1,0)=m(0,0,1)=m(1,1,0)=m(1,0,1)=m(2,0,0)=2,$$

so these are the cases for which we study the evenness conditions. The last column of the following tables indicates which evenness conditions hold. These conditions' holding or not depends only on the number of levels:

(a) For $\sigma^2(1,0,0)$ we have sixteen possibilities according as $a_1(1)$, $a_1(2)$, $a_2(1)$ and $a_3(1)$ are even or odd. In Table 6 we present these possibilities.

Table 6 – Evenness Conditions

$a_1(1)$	$a_1(2)$	$a_2(1)$	$a_3(1)$	$g(1,0,0)$	$g(2,1,0)$	$g(2,0,1)$	$g(1,1,1)$	$g(2,0,0)$	$g(1,1,0)$	$g(1,0,1)$	$g(2,1,1)$	Conditions
Even	Even	Even	Even	Odd	Even	Even	Odd	Even	Odd	Odd	Even	-----
Even	Even	Even	Odd	Odd	Even	Even	Even	Even	Odd	Even	Even	-----
Even	Even	Odd	Even	Odd	Even	Even	Even	Even	Even	Odd	Even	-----
Even	Even	Odd	Odd	Odd	Even	Even	Even	Even	Even	Even	Even	2nd
Even	Odd	Even	Even	Odd	Even	Even	Odd	Even	Odd	Odd	Even	-----
Even	Odd	Even	Odd	Odd	Even	Even	Even	Even	Odd	Even	Even	-----
Even	Odd	Odd	Even	Odd	Even	Even	Even	Even	Even	Odd	Even	-----
Even	Odd	Odd	Odd	Odd	Even	Even	Even	Even	Even	Even	Even	2nd
Odd	Even	Even	Even	Even	Odd	Odd	Even	Odd	Even	Even	Odd	-----
Odd	Even	Even	Odd	Even	Odd	Even	Even	Odd	Even	Even	Even	-----
Odd	Even	Odd	Even	Even	Odd	Odd	Even	Odd	Even	Even	Even	-----
Odd	Even	Odd	Odd	Even	Even	Even	Even	Odd	Even	Even	Even	1st
Odd	Odd	Even	Even	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Even	Odd	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Even	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Odd	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd

(b) For $\sigma^2(0,1,0)$ we have eight possibilities according as $a_1(1)$, $a_2(1)$ and $a_3(1)$ are even or odd. In Table 7 we present these possibilities.

Table 7 – Evenness Conditions

$a_1(1)$	$a_2(1)$	$a_3(1)$	$g(0,1,0)$	$g(1,1,1)$	$g(1,1,0)$	$g(0,1,1)$	Conditions
Even	Even	Even	Odd	Odd	Odd	Odd	-----
Even	Even	Odd	Odd	Even	Odd	Even	-----
Even	Odd	Even	Even	Even	Even	Even	1st, 2nd
Even	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Even	Even	Odd	Even	Even	Odd	-----
Odd	Even	Odd	Odd	Even	Even	Even	2nd
Odd	Odd	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Even	Even	Even	Even	1st, 2nd

(c) For $\sigma^2(0,0,1)$ we have eight possibilities according as $a_1(1)$, $a_2(1)$ and $a_3(1)$ are even or odd. In Table 8 we present these possibilities.

Table 8 – Evenness Conditions

$a_1(1)$	$a_2(1)$	$a_3(1)$	$g(0,0,1)$	$g(1,1,1)$	$g(1,0,1)$	$g(0,1,1)$	Conditions
Even	Even	Even	Odd	Odd	Odd	Odd	-----
Even	Even	Odd	Even	Even	Even	Even	1st, 2nd
Even	Odd	Even	Odd	Even	Odd	Even	-----
Even	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Even	Even	Odd	Even	Even	Odd	-----
Odd	Even	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Even	Odd	Even	Even	Even	2nd
Odd	Odd	Odd	Even	Even	Even	Even	1st, 2nd

(d) For $\sigma^2(1,1,0)$ we have sixteen possibilities according as $a_1(1)$, $a_1(2)$, $a_2(1)$ and $a_3(1)$ are even or odd. In Table 9 we present these possibilities.

Table 9 – Evenness Conditions

$a_1(1)$	$a_1(2)$	$a_2(1)$	$a_3(1)$	$g(1,1,0)$	$g(2,1,1)$	$g(1,1,1)$	$g(2,1,0)$	Conditions
Even	Even	Even	Even	Odd	Even	Odd	Even	-----
Even	Even	Even	Odd	Odd	Even	Even	Even	2nd
Even	Even	Odd	Even	Even	Even	Even	Even	1st, 2nd
Even	Even	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Even	Odd	Even	Even	Odd	Even	Odd	Even	-----
Even	Odd	Even	Odd	Odd	Even	Even	Even	2nd
Even	Odd	Odd	Even	Even	Even	Even	Even	1st, 2nd
Even	Odd	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Even	Even	Even	Even	Odd	Even	Odd	-----
Odd	Even	Even	Odd	Even	Even	Even	Odd	1st
Odd	Even	Odd	Even	Even	Even	Even	Even	1st, 2nd
Odd	Even	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Even	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Odd	Even	Even	Even	Even	1st, 2nd

(e) For $\sigma^2(1,0,1)$ we have sixteen possibilities according as $a_1(1)$, $a_1(2)$, $a_2(1)$ and $a_3(1)$ are even or odd. In Table 10 we present these possibilities.

Table 10 – Evenness Conditions

$a_1(1)$	$a_1(2)$	$a_2(1)$	$a_3(1)$	$g(1,0,1)$	$g(2,1,1)$	$g(1,1,1)$	$g(2,0,1)$	Conditions
Even	Even	Even	Even	Odd	Even	Odd	Even	-----
Even	Even	Even	Odd	Even	Even	Even	Even	1st, 2nd
Even	Even	Odd	Even	Odd	Even	Even	Even	2nd
Even	Even	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Even	Odd	Even	Even	Odd	Even	Odd	Even	-----
Even	Odd	Even	Odd	Even	Even	Even	Even	1st, 2nd
Even	Odd	Odd	Even	Odd	Even	Even	Even	2nd
Even	Odd	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Even	Even	Even	Even	Odd	Even	Odd	-----
Odd	Even	Even	Odd	Even	Even	Even	Even	1st, 2nd

Odd	Even	Odd	Even	Even	Even	Even	Odd	1st
Odd	Even	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Even	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Odd	Even	Even	Even	Even	1st, 2nd

(f) For $\sigma^2(2,0,0)$ we have sixteen possibilities according as $a_1(1)$, $a_1(2)$, $a_2(1)$ and $a_3(1)$ are even or odd. In Table 11 we present these possibilities.

Table 11 – Evenness Conditions

$a_1(1)$	$a_1(2)$	$a_2(1)$	$a_3(1)$	$g(2,0,0)$	$g(2,1,1)$	$g(2,1,0)$	$g(2,0,1)$	Conditions
Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Even	Even	Even	Odd	Even	Even	Even	Even	1st, 2nd
Even	Even	Odd	Even	Even	Even	Even	Even	1st, 2nd
Even	Even	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Even	Odd	Even	Even	Even	Even	Even	Even	1st, 2nd
Even	Odd	Even	Odd	Even	Even	Even	Even	1st, 2nd
Even	Odd	Odd	Even	Even	Even	Even	Even	1st, 2nd
Even	Odd	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Even	Even	Even	Odd	Odd	Odd	Odd	-----
Odd	Even	Even	Odd	Odd	Even	Odd	Even	-----
Odd	Even	Odd	Even	Odd	Even	Even	Odd	-----
Odd	Even	Odd	Odd	Odd	Even	Even	Even	2nd
Odd	Odd	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Even	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Odd	Even	Even	Even	Even	1st, 2nd

3.4. Four factors that cross

Since $L=4$ we write $\sigma^2(h_1,h_2,h_3,h_4)$, $m(h_1,h_2,h_3,h_4)$ and $g(k_1,k_2,k_3,k_4)$. The cases in which $m(h_1,h_2,h_3,h_4)$ is larger than 1 are

$$m(1,0,0,0)=m(0,1,0,0)=m(0,0,1,0)=m(0,0,0,1)=3$$

and

$$m(1,1,0,0)=m(1,0,1,0)=m(1,0,0,1)=m(0,0,1,1)=m(0,1,1,0)=m(0,1,0,1)=2,$$

so these are the cases for which we study the evenness conditions. The last column of the following tables indicates which evenness conditions hold. These conditions' holding or not depends only on the number of levels:

(a) For $\sigma^2(1,0,0,0)$ we have sixteen possibilities according as $a_1(1)$, $a_2(1)$, $a_3(1)$ and $a_4(1)$ are even or odd. In Table 12 we present these possibilities.

Table 12 – Evenness Conditions

$a_1(1)$	$a_2(1)$	$a_3(1)$	$a_4(1)$	$g(1,0,0,0$	$g(1,1,1,0$	$g(1,1,0,1$	$g(1,0,1,1$	$g(1,1,0,0$	$g(1,0,1,0$	$g(1,0,0,1$	$g(1,1,1,1$	Conditions
Even	Even	Even	Even	Odd	Odd	Odd	Odd	Odd	Odd	Odd	Odd	-----
Even	Even	Even	Odd	Odd	Odd	Even	Even	Odd	Odd	Even	Even	-----
Even	Even	Odd	Even	Odd	Even	Odd	Even	Odd	Even	Odd	Even	-----
Even	Even	Odd	Odd	Odd	Even	Even	Even	Odd	Even	Even	Even	-----
Even	Odd	Even	Even	Odd	Even	Even	Odd	Even	Odd	Odd	Even	-----
Even	Odd	Even	Odd	Odd	Even	Even	Even	Even	Odd	Even	Even	-----
Even	Odd	Odd	Even	Odd	Even	Even	Even	Even	Even	Odd	Even	-----
Even	Odd	Odd	Odd	Odd	Even	Even	Even	Even	Even	Even	Even	2nd
Odd	Even	Even	Even	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Even	Even	Odd	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Even	Odd	Even	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Even	Odd	Odd	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Even	Even	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Even	Odd	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Even	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Odd	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd

(b) For $\sigma^2(0,1,0,0)$ we have sixteen possibilities according as $a_1(1)$, $a_2(1)$, $a_3(1)$ and $a_4(1)$ are even or odd. In Table 13 we present these possibilities.

Table 13 – Evenness Conditions

$a_1(1)$	$a_2(1)$	$a_3(1)$	$a_4(1)$	$g(0,1,0,0$	$g(1,1,1,0$	$g(1,1,0,1$	$g(0,1,1,1$	$g(1,1,0,0$	$g(0,1,1,0$	$g(0,1,0,1$	$g(1,1,1,1$	Conditions
Even	Even	Even	Even	Odd	Odd	Odd	Odd	Odd	Odd	Odd	Odd	-----
Even	Even	Even	Odd	Odd	Odd	Even	Even	Odd	Odd	Even	Even	-----
Even	Even	Odd	Even	Odd	Even	Odd	Even	Odd	Even	Odd	Even	-----
Even	Even	Odd	Odd	Odd	Even	Even	Even	Odd	Even	Even	Even	-----
Even	Odd	Even	Even	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Even	Odd	Even	Odd	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Even	Odd	Odd	Even	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Even	Odd	Odd	Odd	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Even	Even	Even	Odd	Even	Even	Odd	Even	Odd	Odd	Even	-----
Odd	Even	Even	Odd	Odd	Even	Even	Even	Even	Odd	Even	Even	-----
Odd	Even	Odd	Even	Odd	Even	Even	Even	Even	Even	Odd	Even	-----
Odd	Even	Odd	Odd	Odd	Even	Even	Even	Even	Even	Even	Even	2nd
Odd	Odd	Even	Even	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Even	Odd	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Even	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Odd	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd

(c) For $\sigma^2(0,0,1,0)$ we have sixteen possibilities according as $a_1(1)$, $a_2(1)$, $a_3(1)$ and $a_4(1)$ are even or odd. In Table 14 we present these possibilities.

Table 14 – Evenness Conditions

$a_1(1)$	$a_2(1)$	$a_3(1)$	$a_4(1)$	$g(0,0,1,0$	$g(1,1,1,0$	$g(1,0,1,1$	$g(0,1,1,1$	$g(1,0,1,0$	$g(0,1,1,0$	$g(0,0,1,1$	$g(1,1,1,1$	Conditions
Even	Even	Even	Even	Odd	Odd	Odd	Odd	Odd	Odd	Odd	Odd	-----
Even	Even	Even	Odd	Odd	Odd	Even	Even	Odd	Odd	Even	Even	-----
Even	Even	Odd	Even	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Even	Even	Odd	Odd	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Even	Odd	Even	Even	Odd	Even	Odd	Even	Odd	Even	Odd	Even	-----
Even	Odd	Even	Odd	Odd	Even	Even	Even	Odd	Even	Even	Even	-----
Even	Odd	Odd	Even	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Even	Odd	Odd	Odd	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Even	Even	Even	Odd	Even	Even	Odd	Even	Odd	Odd	Even	-----
Odd	Even	Even	Odd	Odd	Even	Even	Even	Even	Odd	Even	Even	-----
Odd	Even	Odd	Even	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Even	Odd	Odd	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Even	Even	Odd	Even	Even	Even	Even	Even	Odd	Even	-----
Odd	Odd	Even	Odd	Odd	Even	Even	Even	Even	Even	Even	Even	2nd
Odd	Odd	Odd	Even	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Odd	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd

(d) For $\sigma^2(0,0,0,1)$ we have sixteen possibilities according as $a_1(1)$, $a_2(1)$, $a_3(1)$ and $a_4(1)$ are even or odd. In Table 15 we present these possibilities.

Table 15 – Evenness Conditions

$a_1(1)$	$a_2(1)$	$a_3(1)$	$a_4(1)$	$g(0,0,0,1$	$g(1,1,0,1$	$g(1,0,1,1$	$g(0,1,1,1$	$g(1,0,0,1$	$g(0,1,0,1$	$g(0,0,1,1$	$g(1,1,1,1$	Conditions
Even	Even	Even	Even	Odd	Odd	Odd	Odd	Odd	Odd	Odd	Odd	-----
Even	Even	Even	Odd	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Even	Even	Odd	Even	Odd	Odd	Even	Even	Odd	Odd	Even	Even	-----
Even	Even	Odd	Odd	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Even	Odd	Even	Even	Odd	Even	Odd	Even	Odd	Even	Odd	Even	-----
Even	Odd	Even	Odd	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Even	Odd	Odd	Even	Odd	Even	Even	Even	Odd	Even	Even	Even	-----
Even	Odd	Odd	Odd	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Even	Even	Even	Odd	Even	Even	Odd	Even	Odd	Odd	Even	-----
Odd	Even	Even	Odd	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Even	Odd	Even	Odd	Even	Even	Even	Even	Odd	Even	Even	-----
Odd	Even	Odd	Odd	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Even	Even	Odd	Even	Even	Even	Even	Even	Odd	Even	-----
Odd	Odd	Even	Odd	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Even	Odd	Even	Even	Even	Even	Even	Even	Even	2nd
Odd	Odd	Odd	Odd	Even	Even	Even	Even	Even	Even	Even	Even	1st, 2nd

(e) For $\sigma^2(1,1,0,0)$ we have sixteen possibilities according as $a_1(1)$, $a_2(1)$, $a_3(1)$ and $a_4(1)$ are even or odd. In Table 16 we present these possibilities.

Table 16 – Evenness Conditions

$a_1(1)$	$a_2(1)$	$a_3(1)$	$a_4(1)$	$g(1,1,0,0)$	$g(1,1,1,1)$	$g(1,1,1,0)$	$g(1,1,0,1)$	Conditions
Even	Even	Even	Even	Odd	Odd	Odd	Odd	-----
Even	Even	Even	Odd	Odd	Even	Odd	Even	-----
Even	Even	Odd	Even	Odd	Even	Even	Odd	-----
Even	Even	Odd	Odd	Odd	Even	Even	Even	2nd
Even	Odd	Even	Even	Even	Even	Even	Even	1st, 2nd
Even	Odd	Even	Odd	Even	Even	Even	Even	1st, 2nd
Even	Odd	Odd	Even	Even	Even	Even	Even	1st, 2nd
Even	Odd	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Even	Even	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Even	Odd	Even	Even	Even	Even	Even	1st, 2nd
Odd	Even	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Even	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Odd	Even	Even	Even	Even	1st, 2nd

(f) For $\sigma^2(1,0,1,0)$ we have sixteen possibilities according as $a_1(1)$, $a_2(1)$, $a_3(1)$ and $a_4(1)$ are even or odd. In Table 17 we present these possibilities.

Table 17 – Evenness Conditions

$a_1(1)$	$a_2(1)$	$a_3(1)$	$a_4(1)$	$g(1,0,1,0)$	$g(1,1,1,1)$	$g(1,1,1,0)$	$g(1,0,1,1)$	Conditions
Even	Even	Even	Even	Odd	Odd	Odd	Odd	-----
Even	Even	Even	Odd	Odd	Even	Odd	Even	-----
Even	Even	Odd	Even	Even	Even	Even	Even	1st, 2nd
Even	Even	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Even	Odd	Even	Even	Odd	Even	Even	Odd	-----
Even	Odd	Even	Odd	Odd	Even	Even	Even	2nd

Even	Odd	Odd	Even	Even	Even	Even	Even	1st, 2nd
Even	Odd	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Even	Even	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Even	Odd	Even	Even	Even	Even	Even	1st, 2nd
Odd	Even	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Even	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Odd	Even	Even	Even	Even	1st, 2nd

(g) For $\sigma^2(1,0,0,1)$ we have sixteen possibilities according as $a_1(1)$, $a_2(1)$, $a_3(1)$ and $a_4(1)$ are even or odd. In Table 18 we present these possibilities.

Table 18 – Evenness Conditions

$a_1(1)$	$a_2(1)$	$a_3(1)$	$a_4(1)$	$g(1,0,0,1)$	$g(1,1,1,1)$	$g(1,1,0,1)$	$g(1,0,1,1)$	Conditions
Even	Even	Even	Even	Odd	Odd	Odd	Odd	-----
Even	Even	Even	Odd	Even	Even	Even	Even	1st, 2nd
Even	Even	Odd	Even	Odd	Even	Odd	Even	-----
Even	Even	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Even	Odd	Even	Even	Odd	Even	Even	Odd	-----
Even	Odd	Even	Odd	Even	Even	Even	Even	1st, 2nd
Even	Odd	Odd	Even	Odd	Even	Even	Even	2nd
Even	Odd	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Even	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Even	Even	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Even	Odd	Even	Even	Even	Even	Even	1st, 2nd
Odd	Even	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Even	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Odd	Even	Even	Even	Even	1st, 2nd

(h) For $\sigma^2(0,0,1,1)$ we have sixteen possibilities according as $a_1(1)$, $a_2(1)$, $a_3(1)$ and $a_4(1)$ are even or odd. In Table 19 we present these possibilities.

Table 19 – Evenness Conditions

$a_1(1)$	$a_2(1)$	$a_3(1)$	$a_4(1)$	$g(0,0,1,1)$	$g(1,1,1,1)$	$g(1,0,1,1)$	$g(0,1,1,1)$	Conditions
Even	Even	Even	Even	Odd	Odd	Odd	Odd	-----
Even	Even	Even	Odd	Even	Even	Even	Even	1st, 2nd
Even	Even	Odd	Even	Even	Even	Even	Even	1st, 2nd
Even	Even	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Even	Odd	Even	Even	Odd	Even	Odd	Even	-----
Even	Odd	Even	Odd	Even	Even	Even	Even	1st, 2nd
Even	Odd	Odd	Even	Even	Even	Even	Even	1st, 2nd
Even	Odd	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Even	Even	Even	Odd	Even	Even	Odd	-----
Odd	Even	Even	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Even	Odd	Even	Even	Even	Even	Even	1st, 2nd
Odd	Even	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Even	Even	Odd	Even	Even	Even	2nd
Odd	Odd	Even	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Odd	Even	Even	Even	Even	1st, 2nd

(i) For $\sigma^2(0,1,1,0)$ we have sixteen possibilities according as $a_1(1)$, $a_2(1)$, $a_3(1)$ and $a_4(1)$ are even or odd. In Table 20 we present these possibilities.

Table 20 – Evenness Conditions

$a_1(1)$	$a_2(1)$	$a_3(1)$	$a_4(1)$	$g(0,1,1,0)$	$g(1,1,1,1)$	$g(1,1,1,0)$	$g(0,1,1,1)$	Conditions
Even	Even	Even	Even	Odd	Odd	Odd	Odd	-----
Even	Even	Even	Odd	Odd	Even	Odd	Even	-----
Even	Even	Odd	Even	Even	Even	Even	Even	1st, 2nd
Even	Even	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Even	Odd	Even	Even	Even	Even	Even	Even	1st, 2nd
Even	Odd	Even	Odd	Even	Even	Even	Even	1st, 2nd

Even	Odd	Odd	Even	Even	Even	Even	Even	1st, 2nd
Even	Odd	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Even	Even	Even	Odd	Even	Even	Odd	-----
Odd	Even	Even	Odd	Odd	Even	Even	Even	2nd
Odd	Even	Odd	Even	Even	Even	Even	Even	1st, 2nd
Odd	Even	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Even	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Odd	Even	Even	Even	Even	1st, 2nd

(j) For $\sigma^2(0,1,0,1)$ we have sixteen possibilities according as $a_1(1)$, $a_2(1)$, $a_3(1)$ and $a_4(1)$ are even or odd. In Table 21 we present these possibilities.

Table 21 – Evenness Conditions

$a_1(1)$	$a_2(1)$	$a_3(1)$	$a_4(1)$	$g(0,1,0,1)$	$g(1,1,1,1)$	$g(1,1,0,1)$	$g(0,1,1,1)$	Conditions
Even	Even	Even	Even	Odd	Odd	Odd	Odd	-----
Even	Even	Even	Odd	Even	Even	Even	Even	1st, 2nd
Even	Even	Odd	Even	Odd	Even	Odd	Even	-----
Even	Even	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Even	Odd	Even	Even	Even	Even	Even	Even	1st, 2nd
Even	Odd	Even	Odd	Even	Even	Even	Even	1st, 2nd
Even	Odd	Odd	Even	Even	Even	Even	Even	1st, 2nd
Even	Odd	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Even	Even	Even	Odd	Even	Even	Odd	-----
Odd	Even	Even	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Even	Odd	Even	Odd	Even	Even	Even	2nd
Odd	Even	Odd	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Even	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Even	Odd	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Even	Even	Even	Even	Even	1st, 2nd
Odd	Odd	Odd	Odd	Even	Even	Even	Even	1st, 2nd

4. Final comments

We now can observe that in more then half of the possible combinations of degrees of freedom, at least one of the evenness conditions holds. Therefore the situation in which we have an exact distribution for the generalized F tests is by no means a rare one if there are four factors.

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